

## Partially quenched chiral perturbation theory in the epsilon regime at next-to-leading order

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# Partially quenched chiral perturbation theory in the epsilon regime at next-to-leading order

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**ABSTRACT:** We calculate the partition function of partially quenched chiral perturbation theory in the epsilon regime at next-to-leading order using the supersymmetry method in the formulation without a singlet particle. We include a nonzero imaginary chemical potential and show that the finite-volume corrections to the low-energy constants  $\Sigma$  and  $F$  for the partially quenched partition function, and hence for spectral correlation functions of the Dirac operator, are the same as for the unquenched partition function. We briefly comment on how to minimize these corrections in lattice simulations of QCD. As a side result, we show that the zero-momentum integral in the formulation without a singlet particle agrees with previous results from random matrix theory.

**KEYWORDS:** Lattice QCD, Chiral Lagrangians, QCD

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**1 Introduction**

At low energies, QCD can be described by a chiral effective theory. If the theory is considered in a finite volume and for small quark masses, the standard  $p$ -regime power counting is replaced by the  $\varepsilon$ -regime power counting introduced by Gasser and Leutwyler [1]. To leading order in the  $\varepsilon$ -regime, the partition function is dominated by the contribution of the zero-momentum modes of the Nambu-Goldstone (NG) bosons [1, 2]. In this limit the theory becomes zero-dimensional and is therefore described by chiral random matrix theory (RMT) [3], see [4, 5] for reviews. The low-energy constants (LEC) appearing in the chiral effective Lagrangian, which are of great phenomenological importance, can be determined by fitting analytical results from RMT to lattice data for the eigenvalue spectrum of the Dirac operator. The lowest-order LECs are  $\Sigma$  and  $F$ . While  $\Sigma$  can be determined rather easily, e.g., from the distribution of the small Dirac eigenvalues, the extraction of  $F$  is somewhat more complicated and requires the inclusion of a suitable chemical potential [6, 7].

Since lattice simulations are restricted to a finite volume, it is important to take into account finite-volume corrections to the RMT results, which can be obtained by going to next-to-leading order (NLO) in the  $\varepsilon$ -regime. Recently, finite-volume corrections to the unquenched partition function of QCD in the  $\varepsilon$ -regime were obtained in [8, 9]. However, in order to extract the relevant eigenvalue correlation functions the partially quenched partition function of QCD is needed. A relatively simple method to obtain the partially

quenched theory is to introduce  $n$  replicated flavors in the unquenched theory and then to analytically continue in the discrete number of quark flavors to zero. This so-called replica trick was first used in the theory of disordered systems [10]. It is potentially problematic since the analytic continuation from an isolated set of points is not uniquely defined. Nevertheless, a number of authors have succeeded to construct proper analytic continuations from which correct results could be obtained, see, e.g., [11–13]. Several publications in QCD have used the replica trick for perturbative calculations while borrowing exact result for the non-perturbative part of the theory from RMT [14–17].

In this publication we choose to use an alternative way to obtain the partially quenched theory that does not suffer from the potential problems of the replica trick and can therefore be used to check and extend previous results. In addition to the sea quarks, we introduce fermionic and bosonic valence quarks. In nuclear physics and condensed matter physics this method is known as the supersymmetry method or Efetov method for quenched disorder [18]. In the context of QCD this idea was first used by Morel [19]. The effective low-energy theory of QCD with  $N_f + N_v$  quarks and  $N_v$  bosonic quarks was developed by Bernard and Golterman [20] and by Sharpe and Shoresh [21]. In this work we use the effective theory without a singlet particle as discussed by Sharpe and Shoresh and consider it in a finite volume and for small quark masses. In order to access  $F$  in addition to  $\Sigma$ , we include an imaginary quark chemical potential  $\mu$  [6, 7]. (A first exploratory lattice study of this idea was performed in ref. [22].) We compute the partition function at next-to-leading order in the  $\varepsilon$ -regime and thereby obtain finite-volume corrections of order  $1/\sqrt{V}$  to the partially quenched theory that translate into finite-volume corrections to the LECs  $\Sigma$  and  $F$ . Our results agree with previous results for the unquenched partition function [1, 8, 9]. As a side result we demonstrate that the parametrization of the NG manifold by Sharpe and Shoresh leads to the correct universal limit, in analogy to the results of refs. [23, 24] where a different parametrization was used.

An important question is to what extent the finite-volume effects in the determination of a particular quantity, such as  $\Sigma$  or  $F$ , are universal in the sense that different methods used to determine this quantity give rise to the same finite-volume effects. In general the effects of the finite volume depend on the method, see, e.g., the finite-volume effects in the determination of  $F$  in ref. [25]. In the present paper we show that at next-to-leading order in the  $\varepsilon$ -expansion the partially quenched partition function is equal to its infinite-volume counterpart with  $\Sigma$  and  $F$  replaced by effective values  $\Sigma_{\text{eff}}$  and  $F_{\text{eff}}$ . Since the knowledge of the analytic form of the partially quenched partition function suffices to determine all spectral correlation functions of the Dirac operator  $\mathcal{D}$  we find that all quantities that can be expressed in terms of spectral correlation functions of  $\mathcal{D}$  give rise to the same finite-volume corrections to  $\Sigma$  and  $F$ .

This paper is structured as follows. In section 2 we review the partially quenched theory and how it can be used to compute spectral correlation functions. We also review the corresponding effective low-energy theory in the formulation of Sharpe and Shoresh, both at fixed vacuum angle  $\theta$  and at fixed topology  $\nu$ . In section 3 we compute the finite-volume corrections of order  $1/\sqrt{V}$  to the partially quenched theory, and thus to  $\Sigma$  and  $F$ . We also show that the correct universal limit is obtained from the formulation of Sharpe

and Shores. Conclusions are drawn in section 4. An appendix is provided to collect some useful formulas for the massless propagator in dimensional regularization, including commonly used shape coefficients.

## 2 QCD with $N_f + N_v$ quarks and $N_v$ bosonic quarks in a finite volume

In this section we consider QCD with  $N_f + N_v$  quarks and  $N_v$  bosonic quarks (Morel's bosonic spin-1/2 ghost fields [19]) in a box of volume  $V = L_0 L_1 L_2 L_3$  in the Euclidean formalism. The temporal extent of the box is given by  $L_0$ , and thus the temperature of the system is  $T = 1/L_0$ . Unless stated otherwise we consider the partially quenched case of  $N_f > 0$ .

### 2.1 The partition function and spectral correlation functions

We define QCD with  $N_f + N_v$  quarks and  $N_v$  bosonic quarks by the partition function

$$Z = \int d[A] e^{-S_{\text{YM}}} \left[ \prod_{f=1}^{N_f} \det(\not{D} + m_f) \right] \left[ \prod_{i=1}^{N_v} \frac{\det(\not{D} + m_{vi})}{\det(\not{D} + m'_{vi})} \right], \quad (2.1)$$

where the integral is over all gauge fields  $A$ ,  $S_{\text{YM}}$  is the Yang-Mills action,  $\not{D}$  is the Dirac operator,  $m_1, \dots, m_{N_f}$  are the masses of the sea quarks,  $m_{v1}, \dots, m_{vN_v}$  are the masses of the fermionic valence quarks, and  $m'_{v1}, \dots, m'_{vN_v}$  are the masses of the bosonic valence quarks. By setting the mass  $m_{vi}$  of a valence quark equal to the mass  $m'_{vi}$  of the corresponding bosonic quark, the ratio of determinants of this pair cancels and the flavor  $i$  is quenched.

Next we rewrite the determinants in terms of fermionic quark fields  $\psi$  and bosonic quark fields  $\varphi$  using

$$\det(\not{D} + m) = \int d[\bar{\psi}\psi] e^{-\int d^4x \bar{\psi}(\not{D}+m)\psi} \quad (2.2)$$

and

$$\frac{1}{\det(\not{D} + m)} = \int d[\bar{\varphi}\varphi] e^{-\int d^4x \bar{\varphi}(\not{D}+m)\varphi}, \quad (2.3)$$

where  $\psi$  and  $\bar{\psi}$  are independent Grassmann variables with Berezin integral  $\int d[\bar{\psi}\psi]$ , and  $\varphi$  and  $\bar{\varphi}$  are commuting complex fields related by complex conjugation,  $\bar{\varphi} = \varphi^\dagger$ . The integrals in the exponents are over space-time. Note that the right-hand side of eq. (2.3) only converges if all eigenvalues of  $\not{D}+m$  have a positive real part. Since  $\not{D}$  is anti-Hermitian this condition is satisfied as long as  $\text{Re } m > 0$ . Thus

$$Z = \int d[A] d[\bar{\Psi}\Psi] e^{-S_{\text{YM}} - \int d^4x \bar{\Psi}(\not{D}+M)\Psi} \quad (2.4)$$

with mass matrix  $M = \text{diag}(m_1, \dots, m_{N_f}, m_{v1}, \dots, m_{vN_v}, m'_{v1}, \dots, m'_{vN_v})$  and fields

$$\bar{\Psi} = \left( \bar{\psi} \quad \bar{\varphi} \right), \quad \Psi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix}. \quad (2.5)$$

At nonzero temperature we have to choose anti-periodic boundary conditions in the temporal direction for the fermionic quarks. A pair of fermionic and bosonic quarks at equal mass has to cancel in eq. (2.1), and therefore we have to choose anti-periodic boundary conditions in the temporal direction also for the bosonic quarks (in the same way as Faddeev-Popov ghosts acquire periodic boundary conditions at nonzero temperature [26]). This will amount to periodic boundary conditions for pseudo-NG fermions composed of quarks and bosonic anti-quarks (or of anti-quarks and bosonic quarks).

The vacuum expectation value of an operator  $\mathcal{O}$  is given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int d[A] d[\bar{\Psi}\Psi] \mathcal{O} e^{-S_{\text{YM}} - \int d^4x \bar{\Psi}(\not{D}+M)\Psi}. \quad (2.6)$$

For example, choosing  $N_v = 1$ , the presence of a bosonic quark can be used to obtain the spectral density (or one-point function) of the Dirac operator  $\not{D}$ ,

$$\rho(\lambda) = \langle \text{Tr} \delta(\not{D} - i\lambda) \rangle = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \text{Re} \langle \text{Tr}(\not{D} - i\lambda + \varepsilon)^{-1} \rangle, \quad (2.7)$$

by using

$$\langle \text{Tr}(\not{D} + m)^{-1} \rangle = \frac{\partial}{\partial m_v} \log Z(m_1, \dots, m_{N_f}; m_v, m'_v) \Big|_{m_v=m'_v=m}. \quad (2.8)$$

Analogously, higher-order spectral correlation functions can be obtained using  $N_v = k$ , where  $k$  is the desired order. From these  $k$ -point functions we can also compute individual eigenvalue distributions [27].

## 2.2 The effective low-energy theory at fixed vacuum angle $\theta$

In this section we briefly discuss how to determine the relevant low-energy degrees of freedom for QCD with  $N_f + N_v$  quarks and  $N_v$  bosonic quarks. For details we refer to ref. [21]. The general procedure is as follows. We first determine the non-anomalous symmetries of the Lagrangian that act non-trivially on the vacuum. Then we restrict the remaining symmetry generators to a subset that is sufficient to generate all Ward identities associated with the flavor symmetries. This subset of symmetry generators then determines the relevant NG manifold of the effective low-energy theory. The Lagrangian of the quark sector is given by

$$\mathcal{L}_Q = \bar{\Psi}(\not{D} + M)\Psi, \quad (2.9)$$

which in the massless case ( $M = 0$ ) has vector and axial symmetries. The vacuum of this theory is invariant under the vector symmetry. The axial symmetry, however, acts non-trivially on the vacuum.<sup>1</sup> The axial symmetry is defined by a supermanifold [28] with base

$$\text{Gl}(N_f + N_v) \otimes [\text{Gl}(N_v)/\text{U}(N_v)], \quad (2.10)$$

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<sup>1</sup> For the detailed arguments concerning the symmetry breaking pattern of QCD with  $N_f + N_v$  quarks and  $N_v$  bosonic quarks we again refer to ref. [21].

where  $\text{Gl}$  is the general linear group and  $\text{U}$  is its unitary subgroup. The factor  $\text{Gl}(N_f + N_v)$  acts on the quark sector while  $\text{Gl}(N_v)/\text{U}(N_v)$  acts on the bosonic quark sector [23, 24]. The reason for the smaller symmetry group of the bosonic quark sector is that  $\varphi$  and  $\bar{\varphi}$  are related by complex conjugation, while  $\psi$  and  $\bar{\psi}$  are independent in the functional integral. The measure of the functional integral restricted to the topological sector  $\nu$  transforms under axial transformations  $U_A$  as [29]

$$d[\bar{\Psi}\Psi] \rightarrow \text{Sdet}^\nu(U_A) d[\bar{\Psi}\Psi], \quad (2.11)$$

where  $\text{Sdet}$  is the superdeterminant [18]. Thus, for  $\nu \neq 0$ , only axial transformations with  $\text{Sdet}(U_A) = 1$  leave the measure invariant, i.e., are non-anomalous. Let us express an arbitrary axial transformation  $U_A$  by

$$U_A = \exp(iG_A) = \exp i \begin{pmatrix} u_A & \bar{\kappa}^T \\ \kappa & u'_A \end{pmatrix}, \quad (2.12)$$

where  $\kappa$  and  $\bar{\kappa}$  are independent  $N_v \times (N_f + N_v)$  matrices with elements in the Grassmann algebra,  $u_A$  lives in the group algebra of  $\text{Gl}(N_f + N_v)$ , and  $u'_A$  lives in the group algebra of  $\text{Gl}(N_v)/\text{U}(N_v)$ . The restriction  $\text{Sdet}(U_A) = 1$  amounts to the requirement of a vanishing supertrace [18] of  $G_A$ , i.e.,  $\text{Str} G_A = \text{Tr} u_A - \text{Tr} u'_A = 0$ . Next we restrict the remaining axial symmetries to the minimal subset that is necessary to generate all Ward identities of the full symmetry. Note that  $\text{Gl}(N_f + N_v)$  contains the same generators as  $\text{U}(N_f + N_v)$  with real coordinates replaced by complex ones. Since this does not give rise to additional Ward identities it is sufficient to keep either the real or the imaginary part of each coordinate. The choice made in ref. [21] is

$$G_A = \begin{pmatrix} \pi & \bar{\kappa}^T \\ \kappa & i\pi' \end{pmatrix} + \frac{i\varphi}{\sqrt{(N_f + N_v)N_v N_f}} \begin{pmatrix} N_v \mathbb{1}_{N_f + N_v} & 0 \\ 0 & (N_f + N_v) \mathbb{1}_{N_v} \end{pmatrix}, \quad (2.13)$$

where  $\pi = \pi^\dagger$  and  $\pi' = \pi'^\dagger$  are traceless Hermitian matrices of dimension  $N_f + N_v$  and  $N_v$ , respectively,  $\varphi \in \mathbb{R}$ , and  $\mathbb{1}_n$  is the  $n$ -dimensional identity matrix. This choice leads to the correct signs of the kinetic terms of the NG particles in the effective low-energy theory and will also be used in the rest of this paper. Note that for  $N_f = 0$  also the flavor singlet particle will give rise to long-range correlations [21] and thus has to be included in the effective theory.

The transformation properties of the massive theory under axial transformations as well as the Lorentz group now dictate the form of the Lagrangian of the effective theory [29]. To leading order in  $U(x)$ ,  $\partial_\rho U(x)$ , and  $M$  we find

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4} \text{Str} [\partial_\rho U(x)^{-1} \partial_\rho U(x)] - \frac{\Sigma}{2} \text{Str} [M^\dagger U(x) + U(x)^{-1} M], \quad (2.14)$$

where  $F$  and  $\Sigma$  are low-energy constants and the NG manifold  $U(x)$  is obtained by promoting the coordinates  $\pi$ ,  $\pi'$ ,  $\kappa$ ,  $\bar{\kappa}$ , and  $\varphi$  in eq. (2.13) to fields with

$$U(x) = \exp(iG_A(x)). \quad (2.15)$$

The theory in a  $\theta$ -vacuum is then obtained by rotating the sea quark masses,

$$\mathcal{L}_{\text{eff}}(\theta) = \frac{F^2}{4} \text{Str} [\partial_\rho U(x)^{-1} \partial_\rho U(x)] - \frac{\Sigma}{2} \text{Str} [M^\dagger e^{-i\bar{\theta}/N_f} U(x) + U(x)^{-1} e^{i\bar{\theta}/N_f} M], \quad (2.16)$$

where

$$\bar{\theta} = \theta \begin{pmatrix} \mathbb{1}_{N_f} & 0 \\ 0 & 0 \end{pmatrix} \quad (2.17)$$

is an  $(N_f + 2N_v)$ -dimensional matrix that projects onto the sea-quark sector. The partition function of the effective theory at fixed  $\theta$  is thus given by

$$Z_{\text{eff}}(\theta) = \int d[U] e^{-\int d^4x \mathcal{L}_{\text{eff}}(\theta)}, \quad (2.18)$$

where  $d[U]$  is the invariant integration measure associated with the supermanifold [28]. We restrict ourselves to the effective theory in the rest of this paper and thus drop the subscript in the following.

### 2.3 The effective low-energy theory at fixed topology $\nu$

The partition function at fixed  $\theta$ -angle is given by the Fourier series

$$Z(\theta) = \sum_{\nu=-\infty}^{\infty} e^{i\theta\nu} Z_\nu, \quad (2.19)$$

and thus the partition function at fixed topological charge  $\nu$  is obtained by the Fourier transform

$$Z_\nu = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\theta\nu} Z(\theta). \quad (2.20)$$

For the partition function defined in eq. (2.18) this means

$$Z_\nu = \int d\theta \int d[U] \exp \left\{ -i\theta\nu - \int d^4x \left( \frac{F^2}{4} \text{Str} [\partial_\rho U(x)^{-1} \partial_\rho U(x)] - \frac{\Sigma}{2} \text{Str} [M^\dagger e^{-i\bar{\theta}/N_f} U(x) + U(x)^{-1} e^{i\bar{\theta}/N_f} M] \right) \right\}. \quad (2.21)$$

If we separate the constant mode  $U_0$  from  $U(x)$  by the ansatz

$$U(x) = U_0 \exp(iG_A(x)) \quad (2.22)$$

with  $\int d^4x G_A(x) = 0$  and  $U_0 = \exp(iG_A^0)$ , we can absorb  $\theta$  in  $U_0$  by

$$\pi_0 \rightarrow \tilde{\pi}_0 = \pi_0 - \frac{\theta}{N_f} \begin{pmatrix} \mathbb{1}_{N_f} & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.23)$$



where  $\pi_0$  is the constant mode of the pion fields in the fermionic quark sector of  $G_A^0$ . To avoid confusion with (2.17) we mention that the matrix in (2.23) has dimension  $N_f + N_v$ . Note that we absorb the  $\theta$ -angle only in the sea sector of the theory. This yields

$$Z_\nu = \int d[U] \text{Sdet}^\nu(U_0) \exp \left\{ - \int d^4x \left( \frac{F^2}{4} \text{Str} [\partial_\rho U(x)^{-1} \partial_\rho U(x)] - \frac{\Sigma}{2} \text{Str} [M^\dagger U(x) + U(x)^{-1} M] \right) \right\}, \quad (2.24)$$

where the integration manifold for the constant mode is changed from (2.13) to

$$G_A^0 = \begin{pmatrix} \tilde{\pi}_0 & \bar{\kappa}_0^T \\ \kappa_0 & i\pi'_0 \end{pmatrix} + \frac{i\varphi_0}{\sqrt{(N_f + N_v)N_v N_f}} \begin{pmatrix} N_v \mathbb{1}_{N_f + N_v} & 0 \\ 0 & (N_f + N_v) \mathbb{1}_{N_v} \end{pmatrix}, \quad (2.25)$$

in which  $\tilde{\pi}_0$  now generates  $U(N_f + N_v)$  instead of  $SU(N_f + N_v)$ <sup>2</sup> while  $\pi'_0, \bar{\kappa}_0, \kappa_0$  and  $\varphi_0$  are defined in the same way as their counterparts in eq. (2.13). Note that this parametrization of the constant mode is different from the parametrization used previously in the literature [23, 24]. In section 3.5 we will show that this parametrization again yields the universal RMT result.

### 3 Finite-volume corrections

#### 3.1 The $\varepsilon$ -expansion in the effective theory with imaginary chemical potential

For convenience we redefine the NG manifold with a different normalization of the fields by

$$U(x) = U_0 \exp \left( \frac{i\sqrt{2}}{F} \xi(x) \right) \quad (3.1)$$

with

$$\xi(x) = \begin{pmatrix} \pi(x) & \bar{\kappa}^T(x) \\ \kappa(x) & i\pi'(x) \end{pmatrix} + \frac{i\varphi(x)}{\sqrt{(N_f + N_v)N_v N_f}} \begin{pmatrix} N_v \mathbb{1}_{N_f + N_v} & 0 \\ 0 & (N_f + N_v) \mathbb{1}_{N_v} \end{pmatrix}. \quad (3.2)$$

The constant mode is separated in  $U_0$ , and thus  $\int d^4x \xi(x) = 0$ . For nonzero imaginary chemical potential the Lagrangian of the effective theory is given by

$$\mathcal{L} = \frac{F^2}{4} \text{Str} [\nabla_\rho U(x)^{-1} \nabla_\rho U(x)] - \frac{\Sigma}{2} \text{Str} [M^\dagger U(x) + U(x)^{-1} M] \quad (3.3)$$

with

$$\nabla_\rho U(x) = \partial_\rho U(x) - i\delta_{\rho 0} [C, U(x)], \quad (3.4)$$

where  $C = \text{diag}(\mu_1, \dots, \mu_{N_f}, \mu_{v1}, \dots, \mu_{vN_v}, \mu'_{v1}, \dots, \mu'_{vN_v})$  and  $i\mu_i$  is the imaginary chemical potential of quark flavor  $i$ . We use the  $\varepsilon$ -regime power counting [1] defined by

$$V \sim \varepsilon^{-4}, \quad M \sim \varepsilon^4, \quad \mu \sim \varepsilon^2, \quad \partial_\rho \sim \varepsilon, \quad \xi(x) \sim \varepsilon. \quad (3.5)$$

<sup>2</sup> The addition of  $\mathbb{1}_{N_f}$  to the generators of  $SU(N_f + N_v)$  suffices to generate  $U(N_f + N_v)$ . The normalization of  $\theta$  in eq. (2.23) yields the correct integration domain.

Note that the expansion in  $\varepsilon^2$  amounts to an expansion in  $1/\sqrt{V}$ . To leading order in  $\varepsilon^2$  the Lagrangian is given by

$$\mathcal{L}_0 = \frac{1}{2} \text{Str} [\partial_\rho \xi(x) \partial_\rho \xi(x)] - \frac{\Sigma}{2} \text{Str} [M^\dagger U_0 + U_0^{-1} M] - \frac{F^2}{4} \text{Str} [C, U_0^{-1}] [C, U_0]. \quad (3.6)$$

The next-to-leading order terms in  $\varepsilon^2$  are

$$\mathcal{L}_2 = \mathcal{L}_2^M + \mathcal{L}_2^C + \mathcal{L}_2^N \quad (3.7)$$

with

$$\mathcal{L}_2^M = \frac{\Sigma}{2F^2} \text{Str} [M^\dagger U_0 \xi(x)^2 + \xi(x)^2 U_0^{-1} M], \quad (3.8)$$

$$\mathcal{L}_2^C = -\frac{1}{2} \text{Str} U_0^{-1} C U_0 [\xi(x), [C, \xi(x)]] - \frac{i}{2} \text{Str} (U_0^{-1} C U_0 + C) [\xi(x), \partial_0 \xi(x)], \quad (3.9)$$

$$\mathcal{L}_2^N = \frac{1}{12F^2} \text{Str} [\partial_\rho \xi(x), \xi(x)] [\partial_\rho \xi(x), \xi(x)] - \frac{1}{3\sqrt{2}F} \text{Str} U_0^{-1} [C, U_0] [\xi(x), [\partial_0 \xi(x), \xi(x)]]. \quad (3.10)$$

In this section we will integrate out the fluctuations in  $\xi$  in order to obtain an effective finite-volume partition function. The term  $\mathcal{L}_2^M$  couples to  $U_0$  and  $M$ , and thus corrects the leading-order mass term. In section 3.3 we discuss its effect on the low-energy constant  $\Sigma$ . The term  $\mathcal{L}_2^C$  couples to  $U_0$  and  $C$  and corrects the leading-order chemical potential term. Its effect on the low-energy constant  $F$  is discussed in section 3.4. The first term in  $\mathcal{L}_2^N$  can be ignored since it does not couple to  $U_0$  and therefore only amounts to an overall factor in the effective finite-volume partition function. The second term in  $\mathcal{L}_2^N$  can be ignored at the order at which we are working since it does not give rise to leading-order corrections to  $\Sigma$  or  $F$ .

The integration measure for the parametrization of eq. (3.1) is of the form

$$d[U] = d[U_0] d[\xi] \mathcal{J}(\xi), \quad (3.11)$$

where  $d[U_0]$  is the invariant measure for the constant-mode integral,  $d[\xi]$  is the flat path integral measure of the fields  $\xi$ , and  $\mathcal{J}(\xi)$  is the Jacobian corresponding to the change of variables of eq. (3.1). Since  $\xi$  does not contain constant modes the kinetic term in eq. (3.6) suppresses large fluctuations in  $\xi$ , and thus the integrand vanishes at the integration boundaries of the  $\pi$ - and  $\pi'$ -fields. Therefore the invariant integration measure is well-defined and there are no anomalous contributions by Efetov-Wegner terms [30, 31]. The Jacobian must be of the form

$$\mathcal{J}(\xi) = 1 + \mathcal{O}(\varepsilon^2) \quad (3.12)$$

since there can be no contribution from a linear term in  $\xi$  because of  $\int d^4x \xi(x) = 0$ . Thus, at next-to-leading order the Jacobian only contributes an overall factor to the effective finite-volume partition function.<sup>3</sup>

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<sup>3</sup> At higher orders in  $\varepsilon$  the effects of the Jacobian can no longer be absorbed in an overall prefactor of the partition function.

### 3.2 The propagator

The kinetic term of the Lagrangian in terms of the fields  $\pi$ ,  $\pi'$ ,  $\varphi$ ,  $\bar{\kappa}$ , and  $\kappa$  is given by

$$\begin{aligned} \frac{1}{2} \text{Str} [(\partial_\rho \xi)(\partial_\rho \xi)] &= \frac{1}{2} \text{Tr} [(\partial_\rho \pi)(\partial_\rho \pi)] + \frac{1}{2} \text{Tr} [(\partial_\rho \pi')(\partial_\rho \pi')] + \frac{1}{2} \text{Tr} [(\partial_\rho \varphi)(\partial_\rho \varphi)] \\ &\quad + (\partial_\rho \bar{\kappa}_{ji})(\partial_\rho \kappa_{ji}). \end{aligned} \quad (3.13)$$

Since the mass term  $\mathcal{L}_2^M$  of the Lagrangian, see (3.8), is of order  $\mathcal{O}(\varepsilon^2)$ , the fields are effectively massless. The massless propagator without zero modes,  $\bar{\Delta}(x)$ , is finite in dimensional regularization [32]. In appendix A we give explicit expressions for the relevant propagators used in this work. For the pion fields  $\pi$  and  $\pi'$  the propagators are given by [9, 33]

$$\langle \pi(x)_{ab} \pi(y)_{cd} \rangle_0 = \bar{\Delta}(x-y) \left[ \delta_{ad} \delta_{bc} - \frac{1}{N_f + N_v} \delta_{ab} \delta_{cd} \right], \quad (3.14)$$

$$\langle \pi'(x)_{ab} \pi'(y)_{cd} \rangle_0 = \bar{\Delta}(x-y) \left[ \delta_{ad} \delta_{bc} - \frac{1}{N_v} \delta_{ab} \delta_{cd} \right], \quad (3.15)$$

where the average is defined by

$$\langle \mathcal{O}[\xi] \rangle_0 = \frac{\int d[\xi] \mathcal{O}[\xi] e^{-\int d^4x \mathcal{L}_0}}{\int d[\xi] e^{-\int d^4x \mathcal{L}_0}}. \quad (3.16)$$

For the scalar field  $\varphi$  and for the fermionic field  $\kappa$  the propagators are easily shown to be

$$\langle \bar{\kappa}(x)_{ab} \kappa(y)_{cd} \rangle_0 = -\bar{\Delta}(x-y) \delta_{ac} \delta_{bd}, \quad (3.17)$$

$$\langle \varphi(x) \varphi(y) \rangle_0 = \bar{\Delta}(x-y). \quad (3.18)$$

Using the identities

$$\frac{1}{N_f + N_v} + \frac{N_v^2}{(N_f + N_v) N_f N_v} = \frac{1}{N_f}, \quad (3.19)$$

$$-\frac{1}{N_v} + \frac{(N_f + N_v)^2}{(N_f + N_v) N_f N_v} = \frac{1}{N_f}, \quad (3.20)$$

we thus find the propagator of the composite field  $\xi$  to be

$$\langle \xi(x)_{ab} \xi(y)_{cd} \rangle_0 = \bar{\Delta}(x-y) \left[ \delta_{ad} \delta_{bc} (-1)^{\varepsilon_b} - \frac{1}{N_f} \delta_{ab} \delta_{cd} \right] \quad (3.21)$$

with

$$\varepsilon_b = \begin{cases} 0 & \text{for } 1 \leq b \leq N_f + N_v, \\ 1 & \text{for } N_f + N_v < b \leq N_f + 2N_v. \end{cases} \quad (3.22)$$

Note that there is no explicit dependence on the number  $N_v$  of valence quarks in this propagator.

### 3.3 Finite-volume corrections to $\Sigma$

We now integrate out the fluctuations in the  $\mathcal{O}(\varepsilon^2)$  mass term  $\mathcal{L}_2^M$  to obtain the finite-volume corrections to the leading-order mass term in  $\mathcal{L}_0$ . Using (3.21) it is straightforward to show that

$$\langle \text{Str}[A\xi(x)B\xi(y)] \rangle_0 = \bar{\Delta}(x-y) \left[ \text{Str} A \text{Str} B - \frac{1}{N_f} \text{Str} AB \right]. \quad (3.23)$$

By expanding the action we find that the term

$$\int d^4x \left\langle \frac{\Sigma}{2F^2} \text{Str} [M^\dagger U_0 \xi(x)^2 + \xi(x)^2 U_0^{-1} M] \right\rangle_0 \quad (3.24)$$

corrects the leading-order mass term in the Lagrangian,

$$-\frac{\Sigma}{2} \text{Str} [M^\dagger U_0 + U_0^{-1} M], \quad (3.25)$$

to

$$-\frac{\Sigma}{2} \left[ 1 - \frac{N_f^2 - 1}{N_f F^2} \bar{\Delta}(0) \right] \text{Str} [M^\dagger U_0 + U_0^{-1} M]. \quad (3.26)$$

Thus at next-to-leading order we can read off an effective low-energy constant  $\Sigma_{\text{eff}}$  given by

$$\frac{\Sigma_{\text{eff}}}{\Sigma} = 1 - \frac{N_f^2 - 1}{N_f F^2} \bar{\Delta}(0). \quad (3.27)$$

This is the same result as previously derived for the unquenched partition function [1, 8, 9]. It can be shown that at next-to-next-to-leading order (NNLO) it is no longer possible to absorb the effects of the finite volume in an effective low-energy constant  $\Sigma_{\text{eff}}$ .

### 3.4 Finite-volume corrections to $F$

The calculation of the finite-volume corrections to  $F$  is slightly more involved. The non-vanishing corrections to the leading-order imaginary chemical potential term are given by eq. (3.9). We first calculate the contribution of the first term in (3.9),

$$\begin{aligned} & -\frac{1}{2} \int d^4x \langle \text{Str} U_0^{-1} C U_0 [\xi(x), [C, \xi(x)]] \rangle_0 \\ &= -\frac{1}{2} \int d^4x \langle \text{Str} U_0^{-1} C U_0 [2\xi(x)C\xi(x) - \xi(x)^2 C - C\xi(x)^2] \rangle_0 \\ &= -V \bar{\Delta}(0) [(\text{Str} C)^2 - N_f \text{Str} U_0^{-1} C U_0 C], \end{aligned} \quad (3.28)$$

where we have used (3.23). The first term in (3.28) couples only to  $C^2$  and thus amounts only to a prefactor in the effective finite-volume partition function. The correction to the leading-order Lagrangian obtained from (3.28) is thus given by

$$\frac{\bar{\Delta}(0)}{2} N_f \text{Str} [C, U_0^{-1}] [C, U_0]. \quad (3.29)$$

The contribution of the second term in (3.9) is given by

$$-\frac{i}{2} \int d^4x \langle \text{Str}(U_0^{-1}CU_0 + C)[\xi(x), \partial_0\xi(x)] \rangle_0 \sim \partial_0\bar{\Delta}(0) = 0 \quad (3.30)$$

due to the symmetry  $\bar{\Delta}(x) = \bar{\Delta}(-x)$ . However, the square of this term gives a nonzero contribution. We need to calculate

$$\begin{aligned} & -\frac{1}{2} \left\langle \left( -\frac{i}{2} \int d^4x \text{Str}(U_0^{-1}CU_0 + C)[\xi(x), \partial_0\xi(x)] \right)^2 \right\rangle_0 \\ & = \frac{1}{8} \int d^4x \int d^4y \langle \text{Str}(Y[\xi(x), \partial_0\xi(x)]) \text{Str}(Y[\xi(y), \partial_0\xi(y)]) \rangle_0 \end{aligned} \quad (3.31)$$

with  $Y = U_0^{-1}CU_0 + C$ . After performing all relevant contractions using (3.21) we find

$$\begin{aligned} & \langle \text{Str}[Y\xi(x)\xi(x')] \text{Str}[Y\xi(y)\xi(y')] \rangle_0 \\ & = \bar{\Delta}(x-x')\bar{\Delta}(y-y') \left[ (\text{Str} Y)^2 N_f^2 - 2(\text{Str} Y)^2 + \frac{1}{N_f^2} (\text{Str} Y)^2 \right] \\ & + \bar{\Delta}(x-y)\bar{\Delta}(x'-y') \left[ (\text{Str} Y)^2 - \frac{2}{N_f} \text{Str} Y^2 + \frac{1}{N_f^2} (\text{Str} Y)^2 \right] \\ & + \bar{\Delta}(x-y')\bar{\Delta}(x'-y) \left[ N_f \text{Str} Y^2 - \frac{2}{N_f} \text{Str} Y^2 + \frac{1}{N_f^2} (\text{Str} Y)^2 \right]. \end{aligned} \quad (3.32)$$

Since  $\text{Str} Y = 2 \text{Str} C$  does not couple to  $U_0$  we only need to take into account the terms involving  $\text{Str} Y^2$ . We denote the irrelevant terms by “...” and write

$$(3.32) = -\frac{\text{Str} Y^2}{N_f} [2\bar{\Delta}(x-y)\bar{\Delta}(x'-y') + (2 - N_f^2)\bar{\Delta}(x-y)\bar{\Delta}(x'-y)] + \dots \quad (3.33)$$

We need to calculate

$$\begin{aligned} & (\partial_{x'_0} - \partial_{x_0})(\partial_{y'_0} - \partial_{y_0}) \langle \text{Str}[Y\xi(x)\xi(x')] \text{Str}[Y\xi(y)\xi(y')] \rangle_0 \Big|_{x=x', y=y'} \\ & = -2N_f \text{Str} Y^2 [(\partial_0\bar{\Delta}(x-y))(\partial_0\bar{\Delta}(x-y)) - (\partial_0^2\bar{\Delta}(x-y))\bar{\Delta}(x-y)] + \dots \end{aligned} \quad (3.34)$$

Thus we find

$$(3.31) = -\frac{V}{2} N_f \text{Str} Y^2 \int d^4x (\partial_0\bar{\Delta}(x))^2 + \dots, \quad (3.35)$$

where we have used the fact that the propagator is periodic in time. Therefore the corrections to the effective Lagrangian are given by

$$-\frac{1}{2} N_f \text{Str}[C, U_0^{-1}][C, U_0] \int d^4x (\partial_0\bar{\Delta}(x))^2. \quad (3.36)$$

Combining (3.29) and (3.36), we find that the fluctuations correct the leading-order contribution to the Lagrangian,

$$-\frac{F^2}{4} \text{Str}[C, U_0^{-1}][C, U_0], \quad (3.37)$$

to

$$-\frac{F^2}{4} \text{Str} [C, U_0^{-1}] [C, U_0] \left[ 1 - \frac{2N_f}{F^2} \left( \bar{\Delta}(0) - \int d^4x (\partial_0 \bar{\Delta}(x))^2 \right) \right]. \quad (3.38)$$

Thus at next-to-leading order we find an effective low-energy constant  $F_{\text{eff}}$  given by

$$\frac{F_{\text{eff}}}{F} = 1 - \frac{N_f}{F^2} \left( \bar{\Delta}(0) - \int d^4x (\partial_0 \bar{\Delta}(x))^2 \right). \quad (3.39)$$

This again agrees with the result for the unquenched partition function [8, 9]. As in the case of  $\Sigma$ , at NNLO it is no longer possible to absorb the effects of the finite volume in an effective low-energy constant  $F_{\text{eff}}$ .

### 3.5 The universal limit

In this section we concern ourselves with the limit  $V \rightarrow \infty$  while keeping  $MV\Sigma \sim \mathcal{O}(\varepsilon^0)$ . It is well known that QCD in this limit behaves in a universal way and agrees with chiral RMT. It was first shown in refs. [23, 24] how universal results for the Dirac spectrum (obtained earlier in RMT) can be derived from the effective low-energy theory. In the following we show that the correct universal limit also follows from the effective theory in the formulation of Sharpe and Shoresh described above. For simplicity we restrict ourselves to the case of vanishing imaginary chemical potential,  $C = 0$ .

Since the fluctuations in  $\xi$  are suppressed for  $V \rightarrow \infty$  only the zero-mode integral survives in this limit, and the partition function for fixed topological charge  $\nu$  is given by

$$Z_\nu = \int d[U_0] \text{Sdet}^\nu(U_0) \exp \left( \frac{\Sigma V}{2} \text{Str} [M^\dagger U_0 + U_0^{-1} M] \right), \quad (3.40)$$

where the integration manifold is specified in eq. (2.25). There are different methods to calculate integrals over supermanifolds, see, e.g., [34–36]. In our case it is sufficient to choose an explicit parametrization and reduce the integral to ordinary group integrals. For convenience we use a slightly different notation and calculate

$$Z_\nu = \int d[U] \text{Sdet}^\nu(U) \exp \left( \text{Str} [M^\dagger U + U^{-1} M] \right) \quad (3.41)$$

with integration manifold given by

$$U = \begin{pmatrix} V e^{N_v \varphi} & 0 \\ 0 & V' e^{(N_f + N_v) \varphi} \end{pmatrix} \exp \begin{pmatrix} 0 & \bar{\kappa}^T \\ \kappa & 0 \end{pmatrix} \equiv U_c U_g, \quad (3.42)$$

where  $V \in U(N_f + N_v)$ ,  $V' \in \text{Gl}(N_v)/U(N_v)$  with  $\det V' = 1$ , and  $\varphi \in \mathbb{R}$ . Thus we have

$$\text{Sdet} U = \det V \equiv e^{i\theta} \quad (3.43)$$

with  $\theta \in [0, 2\pi)$ . This is the zero-mode integral following from the parametrization used in the perturbative calculation above. In the literature a similar integral was computed

to determine the static limit of partially quenched chiral perturbation theory [23, 24] that amounts to replacing  $U_c$  by

$$U_c \rightarrow \begin{pmatrix} V & 0 \\ 0 & V' e^{\varphi/N_v} \end{pmatrix}. \quad (3.44)$$

Note first that a parametrization such as  $U = U_c U_g$  above leads to factorization of the corresponding measure as

$$d[U] = d[U_c]d[U_g]. \quad (3.45)$$

This is due to the fact that the invariant length element is

$$\begin{aligned} ds^2 &= \text{Str} [dU d(U^{-1})] \\ &= \text{Str} [dU_c d(U_c^{-1}) + dU_g d(U_g^{-1}) - 2U_c^{-1} dU_c dU_g U_g^{-1}] \\ &= \text{Str} [dU_c d(U_c^{-1}) + dU_g d(U_g^{-1})] \end{aligned} \quad (3.46)$$

since

$$dU_g U_g^{-1} = \begin{pmatrix} 0 & d\bar{\kappa}^T \\ d\kappa & 0 \end{pmatrix}, \quad (3.47)$$

$U_c^{-1} dU_c$  is block diagonal, and therefore

$$\text{Str}[U_c^{-1} dU_c dU_g U_g^{-1}] = 0. \quad (3.48)$$

In both parametrizations the measure of  $V$ ,  $V'$ , and  $\varphi$  also factorizes. Thus

$$d[U] = d[U_g]d[V]d[V']d\varphi \quad (3.49)$$

in both cases. Note that this parametrization has no contributions from Efetov-Wegner terms, as was discussed in a special case in the literature [24]. Introducing the short-hand notation

$$U_g M^\dagger = \begin{pmatrix} X_{\text{ff}} & X_{\text{fb}} \\ X_{\text{bf}} & X_{\text{bb}} \end{pmatrix}, \quad M U_g^{-1} = \begin{pmatrix} Y_{\text{ff}} & Y_{\text{fb}} \\ Y_{\text{bf}} & Y_{\text{bb}} \end{pmatrix}, \quad (3.50)$$

we find for the first parametrization

$$\begin{aligned} \text{Str} [M^\dagger U + M U^{-1}] &= \text{Str} [M^\dagger U_c U_g + M U_g^{-1} U_c^{-1}] = \text{Str} [U_c X + U_c^{-1} Y] \\ &= \text{Tr} [V e^{N_v \varphi} X_{\text{ff}} - V' e^{(N_f + N_v) \varphi} X_{\text{bb}} + V^{-1} e^{-N_v \varphi} Y_{\text{ff}} - V'^{-1} e^{-(N_f + N_v) \varphi} Y_{\text{bb}}]. \end{aligned} \quad (3.51)$$

Next we use a result of [37],

$$\int_{U(p)} d[U] \det^\nu(U) \exp [\text{Tr}(AU + BU^{-1})] = c_p \det(BA^{-1})^{\nu/2} \frac{\det[\mu_i^{j-1} I_{\nu+j-1}(2\mu_i)]}{\Delta(\mu^2)}, \quad (3.52)$$

where  $c_p$  is a constant,  $\Delta(\mu^2)$  is the Vandermonde determinant, and the  $\mu_i^2$  are the eigenvalues of  $AB$ . Thus the integral over  $V$  results in

$$e^{-N_v(N_f+N_v)\nu\varphi} \det(Y_{\text{ff}}X_{\text{ff}}^{-1})^{\nu/2} \frac{\det[\mu_i^{j-1}I_{\nu+j-1}(2\mu_i)]}{\Delta(\mu^2)} \quad (3.53)$$

with  $\mu_i^2$  the eigenvalues of  $X_{\text{ff}}Y_{\text{ff}}$ . In the second parametrization we find

$$\text{Str}[M^\dagger U + MU^{-1}] = \text{Tr}[VX_{\text{ff}} - V'e^{\varphi/N_v}X_{\text{bb}} + V^{-1}Y_{\text{ff}} - V'^{-1}e^{-\varphi/N_v}Y_{\text{bb}}]. \quad (3.54)$$

Note that in this parametrization we also have an additional factor of  $e^{-\varphi\nu}$  from the superdeterminant. Thus the integral over  $V$  leads to

$$e^{-\nu\varphi} \det(Y_{\text{ff}}X_{\text{ff}}^{-1})^{\nu/2} \frac{\det[\mu_i^{j-1}I_{\nu+j-1}(2\mu_i)]}{\Delta(\mu^2)} \quad (3.55)$$

with  $\mu_i^2$  already defined above. Now we let  $\varphi \rightarrow \varphi N_v(N_f + N_v)$  in order to have the same prefactor of  $V'$  and  $V'^{-1}$  in the supertrace. In both parametrizations the resulting integral is

$$\int d[U_g]d[V'] \int_{-\infty}^{\infty} d\varphi e^{-\nu\varphi(N_f+N_v)N_v} \det(Y_{\text{ff}}X_{\text{ff}}^{-1})^{\nu/2} \frac{\det(\mu_i^{j-1}I_{\nu+j-1}(2\mu_i))}{\Delta(\mu^2)} \times \exp\left(-\text{Tr}\left[V'e^{(N_f+N_v)\varphi}X_{\text{bb}} + V'^{-1}e^{-(N_f+N_v)\varphi}Y_{\text{bb}}\right]\right). \quad (3.56)$$

This completes the matching with refs. [23, 24] and is sufficient to show that the parametrization of the NG manifold used in this work leads to the correct universal limit.

In order to extend this proof to the general case of  $C \neq 0$  we would need to calculate the group integral

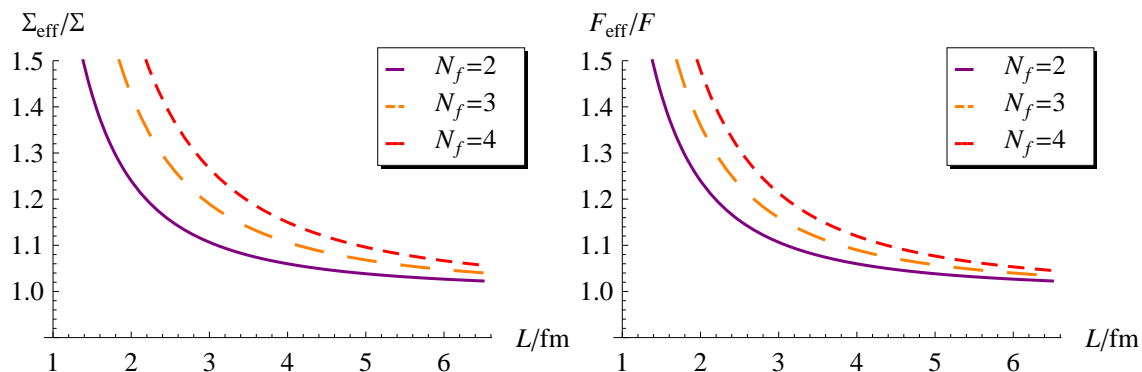
$$\int_{U(p)} d[U] \det^\nu(U) \exp[\text{Tr}(AU + BU^{-1}) + \text{Tr}(DUDU^{-1})], \quad (3.57)$$

where  $A$ ,  $B$ , and  $D$  are arbitrary complex  $p \times p$ -matrices. This, however, is beyond the scope of this work.

## 4 Conclusions

In this work we have calculated the partially quenched partition function of QCD at next-to-leading order in the  $\varepsilon$ -expansion at nonzero imaginary chemical potential. We considered a theory with  $N_f + N_v$  fermionic quarks and  $N_v$  bosonic quarks, as formulated by Sharpe and Shoresh [21], in a finite volume  $V$  with microscopic quark masses  $M$ , i.e.,  $MV\Sigma = \mathcal{O}(\varepsilon^0)$ . The knowledge of the analytic form of the partially quenched partition function suffices to obtain all spectral correlation functions of the Dirac operator  $\not{D}$ . In this sense our results for the finite-volume behavior of the theory hold universally for all observables that can be obtained from spectral correlation functions of  $\not{D}$ . We found that the partially quenched partition function has the same finite-volume corrections as the unquenched





**Figure 1.** Volume-dependence at NLO of the low-energy constants  $\Sigma_{\text{eff}}$  (left) and  $F_{\text{eff}}$  (right) in a symmetric box with dimensions  $L_0 = L_1 = L_2 = L_3 = L$  at  $F = 90$  MeV.

partition function of QCD with  $N_f$  quarks, i.e., at next-to-leading order in  $\varepsilon$  there are effective low-energy constants  $\Sigma_{\text{eff}}$  and  $F_{\text{eff}}$ ,

$$\frac{\Sigma_{\text{eff}}}{\Sigma} = 1 - \frac{N_f^2 - 1}{N_f F^2} \bar{\Delta}(0), \quad (3.27)$$

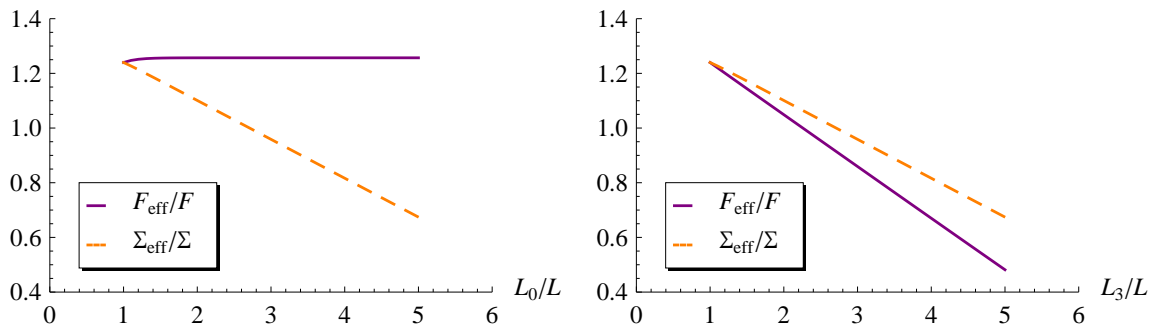
$$\frac{F_{\text{eff}}}{F} = 1 - \frac{N_f}{F^2} \left( \bar{\Delta}(0) - \int d^4x (\partial_0 \bar{\Delta}(x))^2 \right), \quad (3.39)$$

where  $\bar{\Delta}(x)$  is the massless propagator. In appendix A we give closed formulas for the relevant propagators in dimensional regularization and numerical values for typical geometries. As a side result of our calculation we showed that the constant-mode integral of this theory agrees with previous results from random matrix theory. Therefore the correct universal limit is obtained at  $V \rightarrow \infty$ . Note that the proof was only given for vanishing chemical potential and that the knowledge of the group integral (3.57) is needed to complete the proof also for nonzero chemical potential.

In figure 1 we show the finite-volume corrections at NLO to the low-energy constants  $\Sigma$  and  $F$  as a function of the box size  $L$  in a symmetric box. Note that the effects of the finite volume increase with the number of sea quark flavors  $N_f$  and that, depending on  $N_f$ , a box size of 3 – 5 fm is necessary to reduce the effects of the finite volume at NLO to about 10%. The effects are calculated at  $F = 90$  MeV. In figure 2 we show the effect of an asymmetric box with  $N_f = 2$  and  $L = 2$  fm. An important message of this figure is that the magnitude of the finite-volume corrections can be significantly reduced by choosing one large spatial dimension instead of a large temporal dimension. The reason for this behavior is that the chemical potential only affects the temporal direction, see eq. (3.4), and therefore breaks the permutation symmetry of the four dimensions. This manifests itself in the propagator

$$\int d^4x (\partial_0 \bar{\Delta}(x))^2 \quad (4.1)$$

which, as shown in eq. (A.2), contains a term proportional to  $L_0^2/\sqrt{V}$ , where  $L_0$  is the size of the temporal dimension. This term leads to an enhancement of the corrections in case



**Figure 2.** Effect of an asymmetric box with parameters  $N_f = 2$ ,  $L = 2$  fm, and  $F = 90$  MeV. We compare a large temporal dimension  $L_0$  with  $L_1 = L_2 = L_3 = L$  (left) to a large spatial dimension  $L_3$  with  $L_0 = L_1 = L_2 = L$  (right).

of a large temporal dimension. Choosing instead one large spatial dimension, the finite-volume corrections are reduced, unless the asymmetry is too large. For the parameters used in figure 2, the optimal value is  $L_3/L \approx 2$ .

This is good news. Many lattice simulations (at zero chemical potential) are performed with  $L_1 = L_2 = L_3 = L$  and  $L_0 = 2L$ . To determine  $F$ , it suffices to introduce the imaginary chemical potential in the valence sector. Therefore, one can take a suitable set of existing dynamical configurations and redefine  $L_0 \leftrightarrow L_3$  before adding the chemical potential.<sup>4</sup> This will minimize the finite-volume corrections for both  $\Sigma$  and  $F$ , at least for the parameter values chosen in figure 2.

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## A The massless propagator in dimensional regularization

For convenience we collect in this appendix explicit formulas for the massless propagator in dimensional regularization,  $\bar{\Delta}(x)$ , that were derived in refs. [32, 33]. The two relevant quantities for the finite-volume corrections to  $\Sigma$  and  $F$  are given by

$$\bar{\Delta}(0) = -\frac{\beta_1}{\sqrt{V}} \tag{A.1}$$

and

$$\int d^4x (\partial_0 \bar{\Delta}(x))^2 = -\frac{1}{2\sqrt{V}} \left[ \beta_1 - \frac{L_0^2}{\sqrt{V}} k_{00} \right], \tag{A.2}$$

<sup>4</sup> Note that this procedure increases the temperature of the system by a factor of two. One needs to check that the system does not end up in the chirally restored phase, in which our results no longer apply.

$L_0/L$	1	2	3	4
$\beta_1$	0.1404610	0.0836011	-0.0419417	-0.215097
$k_{00}$	0.0702305	0.0833122	0.0833333	0.0833333

**Table 1.** Coefficients for an asymmetric box with  $L_1 = L_2 = L_3 = L$  and temporal dimension  $L_0$ .

$L_3/L$	1	2	3	4
$\beta_1$	0.1404610	0.0836011	-0.0419417	-0.215097
$k_{00}$	0.0702305	-0.0322630	-0.2984300	-0.731240

**Table 2.** Coefficients for an asymmetric box with  $L_0 = L_1 = L_2 = L$  and spatial dimension  $L_3$ . Note that  $\beta_1$  is symmetric under the exchange of the temporal with a spatial dimension.

where  $\beta_1$  and  $k_{00}$  are so-called shape coefficients, i.e., they only depend on the quantities  $l_i = L_i/V^{1/4}$  with  $i = 0, 1, 2, 3$ . The shape coefficient  $\beta_1$  is given by

$$\beta_1 = \frac{1}{4\pi} \left[ 2 - \hat{\alpha}_{-1}(l_j) - \hat{\alpha}_{-1}(l_j^{-1}) \right] \tag{A.3}$$

with

$$\hat{\alpha}_{-1}(x_j) = \int_0^1 dt t^{-2} \left[ \prod_{j=0}^3 S(x_j^2/t) - 1 \right], \tag{A.4}$$

where  $S(x)$  is an elliptic theta-function defined by

$$S(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}. \tag{A.5}$$

The shape coefficient  $k_{00}$  is given by

$$k_{00} = \frac{1}{12} - \sum_{\vec{n}} \frac{1}{4 \sinh(l_0 q_n/2)^2}, \tag{A.6}$$

where the sum is over all integers  $(n_1, n_2, n_3) \neq (0, 0, 0)$  and

$$q_n^2 = \sum_{j=1}^3 (2\pi n_j/l_j)^2. \tag{A.7}$$

In tables 1 and 2 we give numerical values for common shapes.

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